investigation of the recurrence relations for the spheroidal wave functions *

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Abstract

The perturbation method in supersymmetric quantum mechanics (SUSYQM) is used to study the spheroidal wave functions' recurrence relations, which are revealed by the shape-invariance property of the super-potential. The super-potential is expanded by the parameter α and could be gotten by approximation method. Up to the first order, it has the shape-invariance property and the excited spheroidal wave functions are gotten. Also, all the first term eigenfunctions obtained are in closed form. They are advantageous to investigating for involved physical problems of spheroidal wave function.
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1 Introduction of the spheroidal functions

The spheroidal wave equations are extension of the ordinary spherical wave equations. There are many fields where spheroidal functions play important roles just as the spherical functions do. So far, in comparison to simpler spherical special functions (the associated Lengdre's functions) their properties still are difficult for study than their counterpart[1]-[3].

Their differential equations are

$$\left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right]+E+\alpha x^2-\frac{m^2}{1-x^2}\right]\Theta=0 x\in(-1,+1). \tag{1}$$

They only have one more term αX^2 than the spherical ones (the associated Lendgre's equations).

This is a kind of the Sturm-Liouville eigenvalue problem with the natural conditions that Θ is finite at the boundaries $x=\pm1$. The parameter E can only takes the values $E_0,\ E_1,\dots,E_n,\dots$, which are called the eigenvalues of the problem, and the corresponding solutions (the eigenfunctions) $\Theta_0,\Theta_1,\dots,\Theta_n,\dots$ are called the spheroidal wave functions [1]-[3].

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Under the condition $\alpha = 0$, they reduce to the Spherical equation and the solutions to the Sturm-Liouville eigenvalue problem are the associated Legendre-functions $P_n^m(x)$(the spherical functions) with the eigenvalues $E_n = n(n+1),\ n = m+1,\ m+2, \ldots$. Though the spheroidal wave equations are extension of the ordinary spherical wave functions equations, the difference between this two kinds of wave funtions are far greater than their similarity[1].

1. The spherical wave functions $P_n^m(x)$ have many good property such as
   (a) the Legendre-functions $P_n(x) = P_n^0(x)$ are polynomials;
   (b) Back to the variable θ with $x = \cos\theta$, the recursion relation among the Legendre-functions could be written as:
   $$P_n(\cos\theta) \propto \left[-n\cos\theta - \sin\theta\frac{d}{d\theta}\right] P_{n-1}(\cos\theta),\ n = 1, 2, 3, \ldots, \tag{2}$$
   so, all $P_n(\cos\theta)$ could be deduced from the first or ground function P_0 from the recursion relation
   (c) all the associated Legendre-functions $P_n^m(\cos\theta)$ could be derived from P_m^m by
   $$P_n^m(\cos\theta) = \left[-n\cos\theta - \sin\theta\frac{d}{d\theta}\right] P_{n-1}^m(\cos\theta),\ n = m+1, m+2, m+3, \ldots,. \tag{3}$$
2. On the contrary, very little are known about the properties of the spheroidal wave functions:
   (a) All spheroidal wave functions can not be polynomials at all.
   (b) Whether or not there are the similar recurrence relations between them is not clear, at least they have not been found upon to now.

The recursion relations of the spherical wave functions, the associated Legendre functions reflect the the factorization properties of the Legendre equations[16]. This shows that the Legendre equations are solvable. The Super-symmetry quantum mechanics tells that the roots of the solvable properties of differential equations are the super-invariance of their super-potential. In this paper, we testily investigate whether the spheroidal differential equations have the similar shape-invariance property. On accounting of the fact that their differential equations are difficult to treat, we only can rely on the approximate method of small parameter α. The results may make one happy: they have shape-invariance property upon to the first term approximation.

2 Brief review of the theory of SUSY in solving eigenvalue-problem

In recent years, supersymmetric quantum mechanics have attracted tremendous attention for solvable potential problems. They not only provide clear insight into the factorization method of Infeld and Hull [16], but also make great improvement in solving the differential equations. See reference [14] for review on its development.

In supersymmetric quantum mechanics [14]-[15], the Schrödinger equation is

$$H^-\psi^- = -\frac{d^2\psi^-}{dx^2} + V^-(x)\psi^-(x) = E_-\psi^- \tag{4}$$

with $\hbar = 2m = 1$. Here one considers the case of unbroken supersymmetry which demands that the ground state is nodeless with zero energy, and supposes that the superpotential $W(x)$ is continuous and differentiable. The super-potential $W(x)$ satisfies the equation

$$V^-(x) = W^2(x) - W'(x). \tag{5}$$

With the introducing the two operators

$$\mathcal{A} = \frac{d}{dx} + W(x)\ ,\ \mathcal{A}^\dagger = -\frac{d}{dx} + W(x), \tag{6}$$

the corresponding Hamiltonian H^- have a factorized form

$$H^- = \mathcal{A}^\dagger\mathcal{A}\ . \tag{7}$$

The partner potential $V^+(x)$ of V^- is determined by the superpotential $W(x)$ as

$$V^+(x) = W^2(x) + W'(x), \tag{8}$$

and the corresponding Hamiltonian H^+

$$H^+\psi^+ = -\frac{d^2\psi^+}{dx^2} + V^-(x)\psi^+(x) = E_+\psi^+ \tag{9}$$

has a factorized form

$$H^+ = \mathcal{A}\mathcal{A}^\dagger\ . \tag{10}$$

The Hamiltonians H^+ and H^- have exactly the same eigenvalues (or the same spectrum) except that H^- has an additional zero energy and the related eigenstate, that is,

$$E_0^{(-)} = 0\ ,\quad E_{n-1}^{(+)} = E_n^{(-)},\quad \psi_{n-1}^{(+)} \propto \mathcal{A}\,\psi_n^{(-)}\quad ,\quad \mathcal{A}^\dagger\,\psi_n^{(+)} \propto \psi_{n+1}^{(-)}\quad ,\quad n = 1, 2, \ldots . \tag{11}$$

The pair of SUSY partner potentials $V^\pm(x)$ are called shape invariant if they are similar in shape and differ only in the parameters, that is

$$V^+(x; a_1) = V^-(x; a_2) + R(a_1), \tag{12}$$

where a_1 is a set of parameters, a_2 is a function of a_1 (say $a_2 = f(a_1)$) and the remainder $R(a_1)$ is independent of x. One can use the property of shape invariance to obtain the analytic determination of energy eigenvalues and eigenfunctions [14]-[15]. Thus for an unbroken supersymmetry, the eigenstates of the potential $V^-(x)$ are:

$$E_0^- = 0,\quad E_n^- = \sum_{k=1}^{n} R(a_k) \tag{13}$$

$$\Psi_0 \propto \exp\left[-\int_{x_0}^{x} W(y, a_1)dy\right] \tag{14}$$

$$\Psi_n^- = \mathcal{A}^\dagger\,(x, a_1)\Psi_{n-1}^-(x, a_2),\quad n = 1, 2, 3, \ldots \tag{15}$$

Therefore, the shape invariance condition actually is an integrability condition.

3 the eigenvalues and eigenfunctions of the excited states in first order

In the following, we rewrite the equations (1) in the Schrödinger form. Though the form (1) is more familiar for research, the problem is easier to solve in the original differential equation than in the equation (1). The original form is obtained from the eq.(1) by the transformation

$$x = \cos\theta, \tag{16}$$

that is,

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \alpha\cos^2\theta - \frac{m^2}{\sin^2\theta}\right]\Theta = -E\Theta \tag{17}$$

the corresponding boundary conditions become Θ is finite at $\theta = 0,\ \pi$,

From eq.(17) and by the transformation

$$\Theta = \frac{\Psi}{\sin^{\frac{1}{2}}\theta} \tag{18}$$

we could get the Schrödinger form as

$$\frac{d^2\Psi}{d\theta^2} + \left[\frac{1}{4} + \alpha\cos^2\theta - \frac{m^2 - \frac{1}{4}}{\sin^2\theta} + A_s\right]\Psi = 0 \tag{19}$$

and the boundary conditions

$$\Psi|_{\theta=0} = \Psi|_{\theta=\pi} = 0. \tag{20}$$

The equations (19) show the potential is

$$V(\theta,\alpha,m) = -\frac{1}{4} - \alpha\cos^2\theta + \frac{m^2 - \frac{1}{4}}{\sin^2\theta} \tag{21}$$

The super-potential W could be determined by

$$W^2 - W' = V(\theta,\alpha,m) - E_0 = -\frac{1}{4} + \frac{m^2 - \frac{1}{4}}{\sin^2\theta} - \alpha\cos^2\theta - E_0 \tag{22}$$

E_0 is the the first eigenvalue (the ground state energy) of the equations (19). Subtracting E_0 to make it possible to factorize eqns.(19). Actually, it is difficult to find the solutions of the equation (22). So the perturbation methods come naturally, which have been detailed in the reference [18] where the perturbation method in supersymmetry quantum has been used to resolve the ground state function of the spheroidal equations in approximation of little parameter α. Here is brief accounting of the results. One just expands the super-potential W, the central concept of supersymmetry quantum, in the series form of the parameter α. Because the potential is already in the series of α, one could solve the series forms of the super-potential W by solving term by term. Also note the fact the ground energy E_0 must be in the series form of the parameter α and could be obtained by the boundary conditions. So, the super-potential W and E_0 could be expanded as series of the parameter α, that is,

$$W = W_0 + \alpha W_1 + \alpha^2 W_2 + \alpha^3 W_3 + \dots. \tag{23}$$

$$E_0 = \sum_{n=0}^{\infty} E_{0n}\alpha^n \tag{24}$$

The perturbation equation becomes

$$W^2 - W' = V(\theta, \alpha, m) - \sum_{n=0}^{\infty} E_{0n}\alpha^n = -\frac{1}{4} + \frac{m^2 - \frac{1}{4}}{\sin^2\theta} - \alpha\cos^2\theta - \sum_{n=0}^{\infty} E_{0n}\alpha^n \tag{25}$$

There are two lower indices in the parameter E_{0n} with the index 0 refereing to the ground state and the other index n meaning the nth term in parameter α. The last term $\sum_{n=0}^{\infty} 2E_{on}\alpha^n$ is subtracted from the above equation in order to make the ground state energy actually zero for the application of the theory of SUSYQM. Later, one must add the term to our calculated eigen-energy.

In the reference [18], $W_0,\ W_1,\ W_2$ are

$$W_0 = -\left(m + \frac{1}{2}\right)\cot\theta,\ E_{00} = m(m+1). \tag{26}$$

$$W_1 = \frac{\sin\theta\cos\theta}{2m+3} \tag{27}$$

$$W_2 = \left[\frac{-\sin\theta\cos\theta}{(2m+3)^3(2m+5)} + \frac{\sin^3\theta\cos\theta}{(2m+3)^2(2m+5)}\right]. \tag{28}$$

The ground eigenfunction upon to the second order becomes

$$\begin{aligned}
\Psi_0 &= N\exp\left[-\int W d\theta\right] &(29)\\
&= N\exp\left[-\int W_0 d\theta\ - \alpha\int W_1 d\theta - \alpha^2\int W_2 d\theta\right] * \exp O(\alpha^3) &(30)\\
&= (sin\theta)^{m+\frac{1}{2}}\exp\left[-\frac{\alpha\sin^2\theta}{4m+6}\right]\\
&* \exp\left[\frac{\alpha^2\sin^2\theta}{2(2m+3)^3(2m+5)} - \frac{\alpha^2\sin^4\theta}{4(2m+3)^2(2m+5)}\right] * \exp O(\alpha^3). &(31)
\end{aligned}$$

Because the good result of the quantity W_1 in the form of the equation (27), the potential upon to the first order have the property of the shape-invariance. In order to see this property clearly, we rewrite the super-potential W as

$$W = A_1W_0 + \alpha B_1W_1 + O(\alpha^2) \tag{32}$$

then

$$\begin{aligned}
V_-(A_1, B_1, \theta) &= W^2 - W'\\
&= \left[(m+\frac{1}{2})^2A_1^2 - (m+\frac{1}{2})A_1\right]\csc^2\theta - \frac{B_1\alpha}{2m+3}\left[(2m+1)A_1 + 2\right]\cos^2\theta\\
&+ \left[\frac{B_1\alpha}{2m+3} - (m+\frac{1}{2})^2A_1^2\right] + O(\alpha^2) &(33)\\
V_+(A_1, B_1, \theta) &= W^2 + W'
\end{aligned}$$

$$
\begin{aligned}
&= \left[(m+\frac{1}{2})^2A_1^2 + (m+\frac{1}{2})A_1\right]\csc^2\theta - \frac{B_1\alpha}{2m+3}\left[(2m+1)A_1-2\right]\cos^2\theta \\
&+ \left[-\frac{B_1\alpha}{2m+3} - (m+\frac{1}{2})^2A_1^2\right] + O(\alpha^2) \qquad (34)\\
&= V_-(A_2,B_2,\theta) + R(A_1,B_1)) \\
&= \left[(m+\frac{1}{2})^2A_2^2 - (m+\frac{1}{2})A_2\right]\csc^2\theta - \frac{B_2\alpha}{2m+3}\left[(2m+1)A_2+2\right]\cos^2\theta \\
&+ \left[\frac{B_2\alpha}{2m+3} - (m+\frac{1}{2})^2A_2^2\right] + O(\alpha^2) \\
&\quad +R(A_1,B_1) + O(\alpha^2) \qquad (35)
\end{aligned}
$$

where

$$
R(A_1,B_1) = (m+\frac{1}{2})^2A_2^2 - (m+\frac{1}{2})^2A_1^2 - \frac{B_2+B_1}{2m+3}\alpha \tag{36}
$$

with $A_1 = B_1 = 1$. From the above equations, it is easy to get

$$
\begin{aligned}
(m+\frac{1}{2})A_2 &= f_1(A_1) = (m+\frac{1}{2})A_1 + 1 \qquad (37)\\
B_2 &= f_2(A_1,A_2) = \frac{(2m+1)A_1-2}{(2m+1)A_1+4}B_1 \qquad (38)\\
R(A_1,B_1) &= (m+\frac{1}{2})A_1 + 1 - \frac{\alpha}{3}\left(B_2+B_1\right) \qquad (39)
\end{aligned}
$$

Define

$$
\begin{aligned}
A_{k+1} &= f_1(A_k) = A_k + \frac{1}{m+\frac{1}{2}} \qquad (40)\\
B_{k+1} &= f_2(A_k,B_k) = \frac{(2m+1)A_k-2}{(2m+1)A_k+4}B_k \qquad (41)\\
R(A_k,B_k,B_{k+1}) &= (2m+1)A_k + 1 - \frac{\alpha}{3}\left(B_{k+1}+B_k\right), \qquad (42)
\end{aligned}
$$

then the n+1th energy (or the nth excited energy)upon to the first order is

$$
\begin{aligned}
E_n^- &= \sum_{k=1}^{n} R(A_k,B_k,B_{k+1}) \qquad (43)\\
&= \sum_{k=1}^{n}\left((2m+1)A_k\right) - \frac{\alpha}{2m+3}\left(B_{k+1}+B_k\right) \qquad (44)\\
&= n(2m+n+1) - \frac{\alpha}{2m+3}\left[2\left(B_1+B_2+\ldots+B_n\right) - B_1 + B_{n+1}\right]. \qquad (45)
\end{aligned}
$$

By the relation of eq.(41), we get

$$
\left(B_{k+1}+B_k\right) = \frac{2k}{2m+3}\left(B_k - B_{k+1}\right), \tag{46}
$$

therefore,

$$
\left(B_1+B_2+\ldots+B_n\right) = \frac{1}{4}\left[(2m+3)B_1 - (2n+2m+3)B_{n+1}\right]. \tag{47}
$$

We could get B_{n+1} from the relation (41), that is,

$$B_{n+1} = \frac{(2m-1)(2m+1)(2m+3)}{(2n+2m-1)\,(2n+2m+1)\,(2n+2m+3)} B_1, \tag{48}$$

so,

$$\begin{aligned} E_n^- &= -\frac{\alpha}{2(2m+3)}\left[(2m+1)B_1 - (2n+2m+1)\,B_{n+1}\right] + O(\alpha^2) \\ &= n(2m+n+1) \\ &- \frac{\alpha}{2(2m+3)}\left[(2m+1)B_1 - \frac{(2m-1)(2m+1)(2m+3)}{(2n+2m-1)\,(2n+2m+3)}B_1\right] + O(\alpha^2). \end{aligned} \tag{49}$$

As stated in the reference [18], before, the last term in eq.(25) must be added to E_n^- for actual calculation, that is,

$$\begin{aligned} E_n^- + E_{00} + E_{01}\alpha &= n(2m+n+1) + m(m+1) - \frac{\alpha}{2m+3} - \\ &- \frac{\alpha}{2(2m+3)}\left[(2m+1)B_1 - \frac{(2m-1)(2m+1)(2m+3)}{(2n+2m-1)\,(2n+2m+3)}B_1\right] + O(\alpha^2) \end{aligned} \tag{50}$$

$$= n(2m+n+1) + m(m+1) - \frac{\alpha}{2}\left[1 - \frac{(2m-1)(2m+1)}{(2n+2m-1)\,(2n+2m+3)}\right]. \tag{51}$$

Notice that our formula is different from that of the reference [1], the relation is that our $m+n$ is equivalent to l in the reference [1], that is, $n = l - m$ and an overall negative sign between them. Then we can compare the results, which is same:

$$\begin{aligned} E_n^- + E_{00} + E_{01}\alpha + 0(\alpha) &= (l-m)(l+m+1) + m(m+1) - \frac{\alpha}{2}\left[1 - \frac{(2m-1)(2m+1)}{(2l-1)\,(2l+3)}\right] \\ &= l(l+1) - \frac{\alpha}{2}\left[1 - \frac{(2m-1)(2m+1)}{(2l-1)\,(2l+3)}\right]. \end{aligned} \tag{52}$$

The nth excited state upon to the first order is obtained by

$$\psi_n^-(\theta, A_1, B_1) = \mathcal{A}^\dagger \psi_{n-1}^-(\theta, A_2, B_2) \tag{53}$$

$$\propto \mathcal{A}^\dagger(\theta, A_1, B_1)\mathcal{A}^\dagger(\theta, A_2, B_2)\ldots\mathcal{A}^\dagger(\theta, A_n, B_n)\psi_0^-(\theta, A_{n+1}, B_{n+1}), \tag{54}$$

with

$$\psi_0^-(\theta, A_{n+1}, B_{n+1}) = (sin\theta)^{(m+\frac{1}{2})A_{n+1}} \exp\left[\left(-\frac{\alpha B_{n+1}\sin^2\theta}{4m+6}\right)\right] * \exp O(\alpha^2), \tag{55}$$

and

$$\mathcal{A}^\dagger(\theta, A_n, B_n) = -\frac{d}{d\theta} - (m+\frac{1}{2})A_n \cot\theta - \frac{\alpha}{4m+6}B_n \sin 2\theta \tag{56}$$

where $A_n,\ A_{n+1}$ satisfy the relation of (40),(41).

4 conclusion and discussion

In the paper, the first order term of the super-potential obtained has the good property of shape-invariance by the formula of (27), this in turn makes it easy to get the pleasant results of the excited energy and the corresponding excited state functions from the counterparts of the ground state. The results are much more like case of the associated-Lengdre functions. They are new and interesting. The method can also be used to solve higher order terms of the super-potential and the related super-invariance problem of the spin-weighted spheroidal functions. The methods apply to the spin-weighted spheroidal functions for the case $s \neq 0$. These cases have been studied too and will be reported.

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